Parallel Chip Firing Game associated with n-*cube* orientations

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Abstract

We study the cycles generated by the chip firing game associated with n-*cube* orientations. We show the existence of the cycles generated by *parallel evolutions* of even lengths from 2 to 2^n on H_n ($n \geq 1$), and of odd lengths different from 3 and ranging from 1 to $2^{n-1}-1$ on H_n ($n \geq 4$).

Keywords : Graph, chip firing game, parallel evolution, cycle, transient, n-*cube* orientation.

1 Introduction

Consider a digraph $G = (V, A)$, where $V = \{1, ..., n\}$ is the set of vertices and $A \subseteq V \times V$ is the set of arcs. The *out-degree* (resp. *in-degree*) of a vertex i,

hereafter denoted by $d^+(i)$ (resp. $d^-(i)$), is the number of vertices j such that $(i,j) \in A$ (resp. $(j,i) \in A$). A vertex with *out-degree* zero is called a *sink*. All these notions apply to an undirected graph $G = (V, E)$ by considering an edge $e = [i, j]$ as two opposite arcs (i, j) and (j, i).

In the *parallel chip firing game* played on G, a state is a mapping $x : V \to N$ which can be viewed as a distribution of chips onto the vertices of G. A vertex is said to be *active* in a state x if $x(i) \geq d^+(i)$, otherwise it is said to be *passive*. In a move of the game, a state x is transformed into a new state as follows : every vertex tries to send one chip to every *out-neighbor*.

- If it is not possible, i.e. if i is *passive*, then it resigns ;
- Otherwise, vertex i is *active* and sends the chips.

It is easily seen that the number of chips remains constant. Therefore, the evolution is ultimately *periodic*. More precisely, if $x^t, t \geq 0$, denotes the state of the system at time t, then there exists an integer q called *transient length* and another integer p called *period* or *cycle length* such that

$$x^{t+p} = x^t \text{ for } t \geq q, \text{ and } x^{t+p'} \neq x^t \text{ for } p' < p. \tag{1}$$

The sequence $x^0, x^1, ..., x^{q-1}$ is called the *transient* and every sequence of p consecutive states $x^t, x^{t+1}, ..., x^{t+p-1}$, such that $t \geq q$, is called a *cycle* of the evolution.

Following Spencer's introductory paper [SPE 86] which was devoted to the chip firing game on chains, many authors have been interested in this problem. The most interesting questions concern the relationships between the structure of the graph on one hand, and the transients and periods generated by the chip firing game on the other hand. Concerning the *period*, Bitar and Goles have shown that if G is a tree, then only periods one and two occur [BIT 92]. Later, Prisner has studied a generalization of the game by considering multigraphs, i.e.

digraphs with multiplicities on the arcs. He has then shown that there is a sharp contrast in the behavior for eulerian digraphs (i.e. digraphs where the *in-degre* of each vertex equals it *out-degree*). More precisely, he has proved that in every strongly connected euleurian multigraph, any divisor of every dicycle length occurs as a period [PRI 94]. He has also shown that there is no polynomial $h(n)$ such that the periods generated by the chip firing game on digraphs of order n are bounded by $h(n)$. Readers interested by other results on periods and transients of the chip firing game may refer to [TAR 88, AND 89, BIT 89, BJO 91, ERI 91, GOL 93]. Readers interested by combinatorial games may refer to [Gol 02, GM 02, Gol 04, Sjo 05, Fra 09].

There is a particular case where the chip firing game is related to graph orientations. Indeed, let us consider an undirected graph $G = (V, E)$, and let us assume that initially, the edges of G can be oriented in such a way that the number of chips of every vertex equals the *in-degree* of that vertex. If this property is true in the initial configuration, then it remains true throughout the game. One step of the game then consists in reversing the orientations of all edges going into sinks. Goles and Prisner [GOL 00] have studied *gardens of Eden*, i.e. states that can appear only at time $t = 0$. They have also studied the relationships between graph orientations and evolutions induced by states with $|E|$ chips. Moreover, Kiwi, Ndoundam, Tchuente and Goles [KIW 94] have exhibited cycles of exponential length $e^{\Omega(n \log n)}$ generated by the chip firing game associated with the orientations of cascades of rings. Other results on this particular case may be found in [ERI 94].

In this paper, we study the dynamics generated by the chip firing game associated with *n-cube* orientations. More precisely, using a recurrent approach, we show that for $n \geq 4$, there exists cycles of even lengths from 2 to 2^n on H_n ($n \geq 1$), and of odd lengths different from 3 and ranging from 1 to $2^{n-1}-1$ on H_n ($n \geq 4$).

The remainder of this paper is organized as follows. In the next section,

we present some basic notations and definitions related to *n-cubes*. Section 3 is devoted to the recurrent construction of *left cyclic partitions* and possible period lengths whereas section 4 presents some concluding remarks.

2 Basic notations and definitions

An *n-dimensional hypercube* (or *n-cube*) is an undirected graph $H_n = (V, E)$, where $V = \{0,1\}^n$ is the set of vertices and two nodes $u = (u_0, u_1, ..., u_{n-1})$ and $v = (v_0, v_1, ..., v_{n-1})$ are neighbors if and only if they differ in only one bit in their binary representations, i.e. there is an integer i such that $u_i \neq v_i$ and $u_j = v_j$ for $j \neq i$. One can define recursively the *n-cube* as follows :

- The 0-*cube* is reduced to one vertex ;
- H_{n+1} is obtained by taking two copies of H_n and connecting all pairs of equivalent vertices.

Fig. 1 illustrates this constructions for $n = 0, 1, 2, 3$.

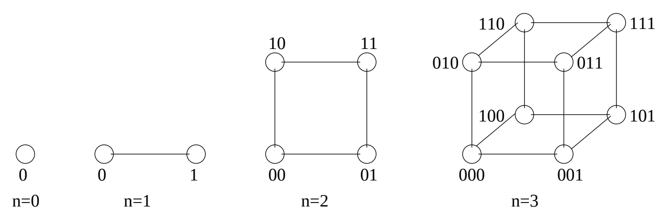


Figure 1: *n-cubes* for $n \leq 3$.

Hereafter, given a set W and a boolean value x, we denote $xW = \{xu : u \in W\}$. With this notation, we can write $H_{n+1} = 0H_n \cup 1H_n$.

Definition 1. A *block-sequential evolution* of the chip firing game associated with graph orientations and played on an *n-cube* is obtained as follows. Consider a sequence of non empty subsets $\{W_i; i \geq 0\}$ of $\{0,1\}^n$. At time t, every vertex u of W_t is considered. If u is a sink then the orientation of all its *in-going* arcs are reversed, otherwise no action is undertaken. Hereafter, we say

that a vertex *fires* at time t if it belongs to W_t and is a *sink* at time t.

The *parallel evolution* is therefore a particular case of the general scheme described above. Another classical evolution scheme is the *sequential evolution* where W_t is reduced to one vertex (i.e. $|W_t| = 1$) and there is a permutation σ of $\{0, ..., 2^n{-}1\}$ such that $\{W_i; i \geq 0\}$ is periodic of period $W_{\sigma(0)}, W_{\sigma(1)}, ..., W_{\sigma(2^n-1)}$. Both parallel and sequential evolutions are particular cases of the so-called *serial-parallel* evolutions where the sequence $\{W_i; i \geq 0\}$ is periodic of period $W_0, W_1, ..., W_{k-1}$, with the constraint that $W_0 \cup W_1 \cup ... \cup W_{k-1}$ is a partition of $\{0, 1, ..., 2^n{-}1\}$.

Definition 2. A partition $S_0 \cup S_1 \cup ... \cup S_{k-1}$ of the vertices of an *n-cube* is called a *left cyclic* partition if the two following statements hold.

- For all i from 0 to $k{-}1$, every vertex of S_i has a neighbor in S_{i-1}, where index operations are performed modulo k.
- For all i from 0 to $k{-}1$, there is no edge between two vertices of S_i.

Comment 2. Canonical decompositions defined in [GOL 00] for acyclic digraphs are obtained from *left cyclic partitions* by orienting the edges such that all arcs from the set S_i go to sets S_j such that $j > i$. On the other hand, *left cyclic partitions* are more restrictive than the partitions introduced in [PRI 94] since we do not allow edges joining two vertices of the same subset. Indeed, in the chip firing game associated with graph orientations, two neighbors cannot fire simultaneously, whereas this situation is possible for the general chip firing game.

We present an important property of left cyclic partitions on an *n-cube*.

Theorem 1 *If a partition $S_0 \cup S_1 \cup ... \cup S_{k-1}$ of the vertices of an n- cube H_n*

is a left cyclic partition then there is a cyclic evolution of the chip firing game associated with graph orientations and played on H_n, such that for every $t \geq 0$, S_t is the set of vertices which are fired at time t.

Proof. Let $S_0, S_1, ..., S_{k-1}$ be a *left cyclic partition.* Consider an orientation where every edge $e = [u, v]$ such that $u \in S_i$ and $v \in S_j, i < j$, is oriented from v to u. It is easily seen that in the parallel chip firing game starting with such a configuration, the subsets of vertices which fire at successive steps correspond to a periodic sequence of period $S_0, S_1, ..., S_{k-1}$.

□

3 Recurrent construction of left cyclic partitions

In this section, we first present the construction of *left cyclic partitions* of even lengths.

Lemma 1 *An n-cube admits left cyclic partitions of all even lengths from 2 to 2^n.*

Proof. Let $H_n = (V, E)$ be an *n-cube* an let p be an even integer between 2 and 2^n. It is well known that, since p is even, there is a cycle $[x_0, x_1, ..., x_{p-1}, x_0]$ of length p in H_n. Now, for every vertex u, let $\Gamma(u)$ denote the set of all neighbors of u in H_n. This notation is naturally extended to a set of vertices. A *left cyclic partition* of order p is obtained as follows.

For $i = 0, ..., p-1$ **do**

  $S_i \leftarrow \{x_i\}$

endfor

$S = V - \{x_0, x_1, ...x_{p-1}\}$

while $(S \neq \emptyset)$ **do**

  For $i \leftarrow 0$ **to** $p-1$ **do**

    $S_{i+1} \leftarrow S_{i+1} \cup (\Gamma(S_i) \cap S)$

$S \leftarrow S - (\Gamma(S_i) \cap S)$

endfor

endwhile

It is obvious that $S_0, ..., S_{p-1}$ is a partition of V and that every vertex in S_i has at least one neighbor in S_{i-1}. So we just need to show that two vertices of the same subset S_i cannot be neighbors. Let a and b be two vertices of S_i.

- There is a path from a to x_0 of length ℓ_1 such that $\ell_1 = i \textbf{ mod } p$,
- There is a path from b to x_0 of length ℓ_2 such that $\ell_2 = i \textbf{ mod } p$,

Since p is even, it follows that $\ell_1 = \ell_2 \textbf{ mod } 2$. Hence, if a and b were neighbors, there would exist a cyclic path of odd length $\ell_1 + \ell_2 + 1$ joining a and b in H_n, which is not possible since H_n is a bipartite graph. This shows that two vertices of the same subset cannot be neighbors.

□.

The following figure displays the partition of order 4 in H_3 obtained by the previous procedure starting with the cycle $[000, 001, 011, 010]$.

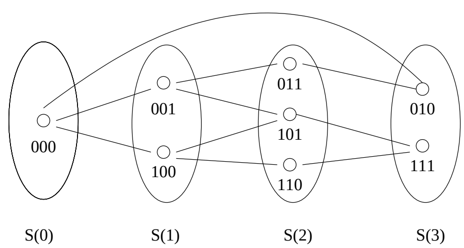


Figure 2: *Left cyclic partition* of orders 4 generated in H_3.

Let us now turn to the construction of *left cyclic partitions* of odd lengths.

Lemma 2 *If S_0, S_1, S_2 is a* left cyclic partition *of $H_n, n \geq 2$, then every vertex of S_i has at least two neighbors in S_{i-1} for $i = 0, 1, 2$.*

Proof. Because of symmetry considerations, we can assume that $i = 2$. So let x be a vertex of S_2. From the definition of *left cyclic partitions*,

- x has a neighbor $x \oplus e_j$ in S_1, where $\oplus$ is the XOR operator and e_j is a

vector of the canonical basis.

- similarly, $x \oplus e_j$ has a neighbor $x \oplus e_j \oplus e_k$ in S_0.

Now consider the vertex $x \oplus e_k$.

- It is a neighbor of x, hence it does not belong to S_2.
- It is a neighbor of $x \oplus e_j \oplus e_k$, hence it does not belong to S_0.

It then follows that $x \oplus e_k$ belongs to S_1, hence x admits two neighbors $x \oplus e_j$ and $x \oplus e_k$ which are both in S_1.

□

Lemma 3 *If* H_n*,* $n \geq 3$ *admits a* left cyclic partition *of order* 3*, then* H_{n-1} *admits a* left cyclic partition *of order* 3*.*

Proof. Let S_0, S_1, S_2 be a *left cyclic partition* of order 3 of H_n. Let x be a vertex of S_i. We can assume without loss of generality that $x = 1a$. Since $H_n = 0H_{n-1} \cup 1H_{n-1}$, $y = 0a$ is the unique neighbor of x in $0H_{n-1}$. Consequently, from lemma 2, x admits a neighbor in $1H_{n-1} \cap S_{i-1}$. This shows that the subgraph $1H_{n-1}$ which is isomorphic to H_{n-1}, contains a *left cyclic partition* of order 3.

□

Proposition 1 *n-cubes do not admit left cyclic partitions of order 3.*

Proof. An *n-cube* with $n \leq 1$ has less than 3 vertices and cannot admit a *left cyclic partition* of order 3. On the other hand, from lemma 2, if S_0, S_1, S_2 is a *left cyclic partition* of an *n-cube*, $n \geq 2$, then every S_i contains at least two elements (i.e. $|S_i| \geq 2$). Consequently, the 2-*cube* H_2 which is of cardinality 4 cannot admit a *left cyclic partition* of order 3. By application of lemma 3, we deduce that no *n-cube*, $n \geq 3$ admits a *left cyclic partition* of order 3.

□

Proposition 1 gives the lower bound for *left cyclic partitions* of odd lengths. Let now study the upper bound.

Proposition 2 *If $S_0, ..., S_{p-1}$ is a left cyclic partition of odd order p of H_n, then $p \leq 2^{n-1} - 1$.*

Proof. We just have to show that in such a case, $|S_i| \geq 2$ for $i = 0, ..., p-1$. Indeed, starting from a vertex $a_{p-1} \in S_{p-1}$, we construct a chain [a_{p-1},a_{p-2},... , a_0, b_{p-1}, b_{p-2}, ..., b_0] such that $a_i, b_i \in S_i$ for $i = 0, ..., p-1$. It is clear that $a_i \neq b_i, i = 0, ..., p-1$, otherwise we would have displayed a closed path of odd length in H_n which is not possible.

□

Now that we have established lower and upper bounds for *left cyclic partitions* of odd lengths, let us show that all intermediate lengths are admissible.

Lemma 4 *If H_n admits a* left cyclic partition *of order p, then H_{n+1} admits left cyclic partition of order p.*

Proof. If $S_0, ..., S_{p-1}$ is a left cyclic partition of order p in H_n, then it is easily checked that $1S_i \cup 0S_{i-1}, i = 0, ..., p-1$ is a *left cyclic partition* of order p in H_{n+1}

□.

Lemma 5 *If H_n admits a* left cyclic partition *of odd order p, $p \geq 5$ then H_{n+1} admits a* left cyclic partition *of order $2p-1$. Moreover, if $p \geq 7$, then H_{n+1} admits a left cyclic partition of order $2p-3$.*

Proof. Let $S_0, S_1, ..., S_{p-1}$ be a *left cyclic partition* of odd order p.

- Case $p \geq 5$

The following sequence is a *left cyclic partition* of order $2p-1$ in H_{n+1}.
$0S_0$, $1S_0 \cup 0S_1$, $1S_1$, $1S_2$, $0S_2$, $0S_3$, $1S_3$ ..., $1S_{2i}$, $0S_{2i}$, $0S_{2i+1}$, $1S_{2i+1}$,..., $1S_{p-3}$, $0S_{p-3}$, $0S_{p-2}$, $1S_{p-2}$, $1S_{p-1}$, $0S_{p-1}$,

- Case $p \geq 7$

A *left cyclic partition* of order $2p-3$ in H_{n+1} is obtained from the *left cyclic*

partition exhibited in the case $p \geq 5$ by replacing the subsequence $1S_2$, $0S_2$, $0S_3$, $1S_3$, $1S_4$, $0S_4$, $0S_5$, $1S_5$ by $1S_2$, $0S_2 \cup 1S_3$, $0S_3$, $0S_4$, $1S_4 \cup 0S_5$, $1S_5$.

□

Lemma 6 *H_4 admits left cyclic partitions of orders 5 and 7.*

Proof.

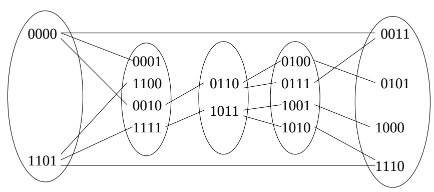
• A *left cyclic partition* of order 5 in H_4 is the following :

$\{0000, 1101\}$, $\{0001, 1100, 0010, 1111\}$, $\{0110, 1011\}$, $\{0100, 0111, 1001, 1010\}$, $\{0011, 0101, 1000, 1110\}$.

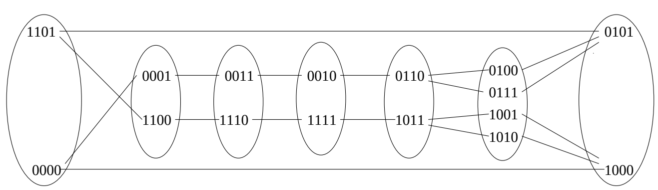
• A *left cyclic partition* of order 7 in H_4 is the following :

$\{0000, 1101\}$, $\{0001, 1100\}$, $\{0011, 1110\}$, $\{0010, 1111\}$, $\{0110, 1011\}$, $\{0100, 0111, 1001, 1010\}$, $\{0101, 1000\}$.

Fig. 3 displays the partitions.



Left cyclic partition of order 5 in H4



Left cyclic partition of order 7 in H4

Figure 3: *Left cyclic partitions* of orders 5 and 7 in H_4.

□

Lemma 7 *H_n, $n \geq 4$, admits a* left cyclic *partition of order* $2^{n-1} - 1$.

Proof. Consider the sequence $\{u_i; 0 \leq i \leq 2^{n-1}-1\}$, defined by $u_i = bin(i) \oplus bin(i/2)$, where $bin(x)$ is the n-position binary representation of the integer

x, and symbol / denotes integer division. It can be easilly checked that this sequence corresponds to a *hamiltonian cycle* in H_{n-1}. Now, let us denote $v_i = u_i \oplus 1 \oplus 2^{n-2}$ (i.e. v_i is obtained from u_i by changing the first and last bits) and $N = 2^n$. It is also easy to check that $\{v_i; 0 \leq i \leq 2^{n-1} - 1\}$ is a *hamiltonian cycle* of H_{n-1}. Let us now consider the following sets

$$\{0u_0, 1v_0\}, ..., \{0u_{N-4}, 1v_{N-4}\}, \{0u_{N-3}, 0u_{N-1}, 1v_{N-3}, 1v_{N-1}\}, \{0u_{N-2}, 1v_{N-2}\}. \tag{2}$$

At this step, it is important to recall that two vertices referenced by i and j are neighbors in the hypercube if and only if there is an integer k such that $i \oplus j = 2^k$. Observe that $0u_i \oplus 1v_i = 2^{n-1} \oplus (u_i \oplus v_i) = 2^{n-1} \oplus 1 \oplus 2^{n-2}$. Hence, $0u_i$ and $1v_i$ are not neighbors in the hypercube H_n . On the other hand, $u_{N-4} = 100...010$, $u_{N-2} = 10...01$, $u_{N-1} = 10...0$ and $v_0 = u_0 \oplus 1 \oplus 2^{n-2} = 10...01 = u_{N-2}$. Hence,

$$0u_{N-4}, 0u_{N-1}, 0u_{N-2}, 1v_0 \text{ is a chain of } H_n. \tag{3}$$

Moreover, $v_{N-4} = 0...011$, $v_{N-2} = 0...0 = u_0$ and $v_{N-1} = 0...01 = u_1$. Hence

$$1v_{N-4}, 1v_{N-1}, 1v_{N-2}, 0u_0 \text{ is a chain of } H_n. \tag{4}$$

Properties 3 and 4 together with the fact that $\{u_i; 0 \leq i \leq 2^{n-1}-1\}$ and $\{v_i; 0 \leq i \leq 2^{n-1}-1\}$ are both *hamiltonian cycles* of H_n, imply that the partition exhibited in 2 is a *left cyclic partition.*

□

Proposition 3 *H_n, $n \geq 4$, admits left cyclic partitions of all odd orders from 5 to $2^{n-1} - 1$.*

Proof. We proceed by induction on n. For $n = 4$ the result follows from lemma 6.

Assuming that the result holds for $n \geq 4$, let us consider an $(n+1)$-*cube* together with an odd integer $p \in [5, 2^n - 1]$.

• Case 1 : $5 \leq p \leq 2^{n-1} - 1$. The result follows from the induction hypothesis by application of lemma 4.

• Case 2 : $2^{n-1} - 1 < p < 2^n - 1$. There is an odd integer q, $7 < q < 2^{n-1} - 1$, such that $p = 2q - 1$ or $p = 2q - 3$. The result follows from the induction hypothesis by application of lemma 5.

• Case 3 : $p = 2^n - 1$. The result follows from lemma 7.

□

We are now ready to state the main theorem.

Theorem 2 *There exists cycles generated by the parallel chip firing game associated with n-cube orientations, $n \geq 4$, are of even lengths from 2 to 2^n, and of odd lengths different from 3 and ranging from 1 to $2^{n-1} - 1$.*

Proof. We just need to show that this property holds for *left cyclic partitions* of vertices of *n-cubes*. The existence of *left cyclic partitions* of all even lengths from 2 to 2^n follows from lemma 1. Let us turn to odd periods p.

• Case 1 : $p = 1$. Consider and orientation which contains a *hamiltonian cycle*. Clearly, such a configuration is a *fixed point* for the chip firing game associated with graph orientations.

• Case 2 : $p = 3$. The non existence of period 3 follows from proposition 1.

• Case 3 : $5 \leq p \leq 2^{n-1} - 1$. The existence of this period follows from proposition 3.

□

4 Conclusion

We show in the particular case of *parallel evolutions* on *n-cube*, the existence of cycles of even lengths from 2 to 2^n, and of odd lengths different from 3 and ranging from 1 to $-1+2^{n-1}$. In case of *parallel evolutions* on *n-cube*, the existence of cycles of lengths greater than 2^n remains an open question.

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